Ministry of Science and Higher Education of the Russian Federation

Ryazan Institute (branch) of
the Federal State Educational Institution of Higher Education
«Moscow Polytechnic University»

Department of «Informatics and Information Technologies»

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LINEAR ALGEBRA,
VECTOR ALGEBRA AND ANALYTICAL GEOMETRY

WorkBook

Ryazan

2021

UDC 517
LBC 22.11
A 35

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A 35 Linear algebra, vector algebra and analytical geometry. Work-Book/I.A. Azizyan. – Ryazan: Ryazan Institute (branch) of Moscow Polytechnic University, 2021. – 32 p.

This workbook consists of 3 parts devoted to the mathematical methods of linear algebra, vector algebra and analytical geometry based on the vector analysis technique. The basic concepts are explained by examples andillustrated by figures.

The workbook is helpful for students who want to understand and be able to use matrix operations, solve systems of linear equations, analyze relative positions of figures, transform coordinate systems, and so on.

Published by the decision of the methodological council of the Ryazan Institute (branch) Moscow Polytechnic University.

UDC 517
LBC 22.11

Contents

1 Linear algebra

A **matrix** is a rectangular array of numbers, algebraic symbols or mathematical functions, provided that such arrays are added and multiplied according to certain rules.

Matrices are denoted by upper case letters: *A*, *B*, *C*, …

The size of a matrix is given by the number of rows and the number of columns.

A matrix with *m* rows and *n* columns is called an $m \times n$ matrix (pronounce *m*-by-*n* matrix).

The numbers *m* and *n* are the dimensions of the matrix. Two matrices have the same size, if their dimensions are equal.

Members of a matrix are called its **matrix elements** or **entries**.

A matrix with one row is called a **row matrix.** A matrix with one column is called a **column matrix**.

A **square matrix** has as many rows as columns, the number of which determines the order of the matrix, that is, an $n \times n$ matrix is the matrix of the n-th **order**.

In the general form, a matrix is written as follows:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

A short form of this equality is $A = \|a_{ij}\|$.

1.1 Matrix operations

Two matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$, are equal, if they have the same sizes and their elements are equal by pairs, that is $A = B \leftrightarrow a_{ij} = b_{ij}$ for each pair of indexes $\{i, j\}$.

Any matrix *A* may be multiplied on the right or left by a scalar quantity λ.

The product is the matrix $B = \mu A$ (of the same size as *A*) such that $b_{ij} = \mu a_{ij}$ for each pair of indexes $\{i, j\}$.

To multiply a matrix by a scalar, multiply every matrix element by that scalar.

To add matrices, add the corresponding matrix elements.

If $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ are matrices of the same size, then the sum $A + B$, is the matrix $C = \|c_{ij}\|$ such that $c_{ij} = a_{ij} + b_{ij}$ for each pair of indexes $\{i, j\}$.

Multiplication of a Row by a Column. Let A be a row matrix having as many elements as a column matrix B. In order to multiply A by B, it is necessary to multiply the corresponding elements of the matrices and to add up the products.

Symbolically, $A \cdot B = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1}$.

Matrix multiplication. The product of two matrices, A and B, is defined, if and only if the number of elements in a row of A equals the number of ones in a column of *B*. Let *A* be an $m \times l$ matrix and *B* be an $l \times n$ matrix. Then the product *AB* is the $m \times n$ matrix such that its entry in the *i*-th row and the *j*-th column is equal to the product of the *i*-th row of *A* and the *j*-th column of *B*.

Properties involving Addition

1) For any matrix *A* there exists the opposite matrix (– *A*) such that

$$A + (-A) = A - A = 0.$$

2) If *A* and *B* are matrices of the same size, then

$$A + B = B + A.$$

3) If *A*, *B*, and *C* are matrices of the same size, then

$$(A + B) + C = A + (B + C).$$

4) The transpose of the matrix sum is the sum of the transpose of the matrices:

$$(A+B)^T = A^T + B^T.$$

Properties involving Multiplication

1) Let *A* be a matrix. If α, β are scalar quantities, then $\alpha(\beta A) = (\alpha\beta)A$.
2) Let *A* and *B* be two matrices such that the product *AB* is defined. If α is a scalar quantity, then $\alpha(AB) = (\alpha A)B$.
3) Let *A*, *B* and *C* be three matrices such that all necessary multiplications are appropriate. Then $A(BC) = (AB)C$.
4) Let *A* and *B* be two matrices such that the product *AB* is defined. Then $(A \cdot B)^T = B^T \cdot A^T$.
5) If *A* and *B* are diagonal matrices of the same order, then $AB = BA$.

Problem 1. Which of the below matrices are equal, if any?

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 4 & 5 & 0 \\ 1 & 2 & 0 \end{pmatrix}, D = \begin{pmatrix} 5-1 & \sqrt{25} \\ sin 90° & 2 \end{pmatrix}.$$

Problem 2. Given the matrices A, B, C, find the linear combination $2A - 3B + 4C$.

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 10 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & -2 \\ 4 & 1 \end{pmatrix}.$$

Problem 3. Express matrix $A = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ as the linear combination of the matrices

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 4. Determine which of the matrix products *AB* and *BA* are defined. If the product is appropriate, find the size of the matrix obtained.

1) *A* is a $3 * 5$ matrix and *B* is a $5 * 2$ matrix;

2) A is a $3 * 2$ matrix and B is a $2 * 3$ matrix;
3) A is a $4 * 2$ matrix and B is a $4 * 2$ matrix;
4) A is a $1 * 7$ matrix and B is a $7 * 1$ matrix;
5) A and B are square matrices of the fifth order.

Problem 5. Let $A = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 2 \end{pmatrix}$.

Evaluate the matrix products AA^T and A^TA. If the difference $AA^T - A^TA$ is defined, find it. If not, explain why.

Problem 6. Find the matrix product AB, if $A = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -2 & 1 \\ -3 & 5 & -1 \\ 4 & -3 & 2 \end{pmatrix}$.

Problem 7. Given three matrices A= $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 8 & 1 \end{pmatrix}$, find the matrix products $(AB)C$ and $A(BC)$.

Problem 8. Let A and B be two diagonal matrices of the order 4. Find the matrix $AB - BA$.

Problem 9. Find A^{50}, if $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Problem 10. Let $A = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 4 & -1 \\ 5 & -3 & 7 \end{pmatrix}$. Find a matrix B such that $C = A + B$ is a diagonal matrix.

Problem 11. Which of the below matrices are symmetric? Which are skew-symmetric?

$$A = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ -1 & 1 & 7 \end{pmatrix}, B = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 3 & 2 & 5 \\ 2 & -4 & 1 \\ 5 & 1 & 7 \end{pmatrix}.$$

Problem 12. Let $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$ and $f(x) = 3x^2 + 5x - 4$. Find f(A).

1.2 Determinants

A matrix of the first order contains only one element.

The determinant $detA = a_{11}$.

Let A be a square matrix of the second order: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

$detA = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$.

If a matrix has the third then we have to consider all possible permutations. To remember this formula, apply the Sarrus Rule which is shown in thefigure below

$+a_{1,1}a_{2,2}a_{3,3}$	$-a_{1,3}a_{2,2}a_{3,1}$
$+a_{1,2}a_{2,3}a_{3,1}$	$-a_{1,2}a_{2,1}a_{3,3}$
$+a_{1,3}a_{2,1}a_{3,2}$	$-a_{1,1}a_{2,3}a_{3,2}$

Figure 1 ─ Basic formulas

The elements on a diagonal or at the vertices of a triangular form the product of three elements. If the base of the triangle is parallel to the leading diagonal of the matrix, the product keeps the sign; otherwise, the product changes the sign.

Evaluation of determinants by elementary operations on matrices By means of elementary row and column operations, a matrix can be reduced to the triangular form, the determinant of which is equal to the product of the diagonal elements.

Let us define the **elementary operations**.

In view of the properties of determinants, any techniques which are developed for rows may be also applied to columns.

In order to calculate a determinant one may:

1) Interchange two rows. As a result, the determinant changes its sign.
2) Multiply a row by a nonzero number.

As a consequence of this operation, the determinant is multiplied bythat number.

3) Add a row multiplied by a number to another row.

By this operation, the determinant holds its value.

We can also use the elementary operations to get some row or column consisting of zero elements except for one element, and then to expand the determinant by that row (or column).

Problem 1. Apply the Sarrus Rule to calculate the determinants of thematrices

$$A=\begin{pmatrix}0 & -2 & 1\\ 2 & 4 & -1\\ -1 & 1 & 7\end{pmatrix}, B=\begin{pmatrix}0 & -2 & 1\\ 2 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}, C=\begin{pmatrix}3 & 2 & 5\\ 2 & -4 & 1\\ 5 & 1 & 7\end{pmatrix}.$$

Problem 2. Let $A=\begin{pmatrix}3 & -2\\ -1 & 4\end{pmatrix}$ and $B=\begin{pmatrix}6 & 3\\ 5 & 2\end{pmatrix}$. Find the determinant of the matrix product A^2B^3.

Problem 3. Evaluate $detA^{10}$, if $A=\begin{pmatrix}4 & 2\\ 5 & 3\end{pmatrix}$.

Problem 4. Let A be a square matrix of the third order such that $detA=5$.

Find $\det(2A)$; 2) $detA^T$; 3) $detA^{3T}$; 4) $\det(2A^2A^T)$.

Problem 5. Let $A = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 4 & -1 \\ 5 & -3 & 7 \end{pmatrix}$.

Evaluate $detA$ using expansion according to the second column. The expand the determinant by the first row and compare the results obtained.

Problem 6. By elementary row and column operations, reduce the matrix $A = \begin{pmatrix} 2 & 3 & -2 \\ 4 & 0 & 7 \\ -1 & 2 & 3 \end{pmatrix}$ to the triangular form and calculate $detA$.

Problem 7. Let $A = \begin{pmatrix} 5 & 1 & -2 \\ 1 & 3 & 6 \\ 0 & 7 & 1 \end{pmatrix}$. Calculate $detA$.

1.3 Inverse matrices

In a square matrix $A = \|a_{ij}\|$ the elementsa_{ii} are called the **diagonal matrix elements**. The set of the entries a_{ii} forms the leading (or principle) diagonal of the matrix.

A square matrix $A = \|a_{ij}\|$ is called a **diagonal matrix**, if off-diagonal elements are equal to zero.

A matrix is called a **zero-matrix** (0-matrix), if it consists of only zero elements: $a_{ij=0}$

A square matrix has a **triangular form**, if all its elements above or below the leading diagonal are zeros; all $a_{ij} = 0$ for $i > j$ or for $i < j$.

Given an $m \times n$ matrix $A = \|a_{ij}\|$, the **transpose of *A*** is the $n \times m$ matrix $A^T = \|a_{ji}\|$ obtained from *A* by interchanging its rows and columns. This means that the rows of the matrix *A* are the columns of the matrixA^T.

A square matrix $A = \|a_{ij}\|$, is called a **symmetric** matrix, if *A* is equal to the transpose of *A*: $A = A^T$.

A square matrix $A = \|a_{ij}\|$, is called a **skew-symmetric** matrix, if *A* is equal to the opposite of its transpose: $a_{ij} = -a_{ji}$.

For any regular matrix A there exists the unique inverse matrix: $A^{-1} = \frac{1}{\det A} adjA$. Any singular matrix has no an inverse matrix.

Gaussian Elimination. Consider the augmented matrix of system

$$(A|B) = \left(\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}\middle| \begin{matrix} b_1 \\ \ldots \\ b_m \end{matrix}\right).$$

There is one-to-one correspondence between the elementary transformations of the linear system and linear row operations on the augmented matrix.

Interchanging two equations of the system corresponds to interchanging the rows of the augmented matrix.

Multiplication of an equation by a nonzero number corresponds tomultiplication of the row by that number.

Addition of two equations of the system corresponds to addition ofthe rows of the matrix.

The main idea is the following.

First, transform the augmented matrix to the upper triangle form or row echelon form:

$$(A|B) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \dots \\ 0 & \cdots & a_{mn} & b_m \end{array}\right).$$

Then write down the linear system corresponding to the augmented matrix in the triangle form or reduced row echelon form. This system is equivalentto the given system but it has a simpler form.

Finally, solve the system obtained by the method of back substitution. If itis necessary, assign parametric values to some unknowns.

This systematic procedure of solving systems of linear equations by elementary row operations is known as **Gaussian elimination**.

Cramer's Rule: Let $AX = B$ be a system of *n* linear equations with *n* unknowns.

If the coefficient matrix *A* is regular, then the system is consistent and has a unique solution set $\{x_1, x_2, \dots, x_n\}$ which is represented by the formula: $x_i = \frac{\Delta x_i}{\Delta}$, Δx_i is the determinant of the matrix obtained by replacing the *i*-th column of *A* with the column matrix *B*. $\Delta = detA$.

Problem 1. Find the cofactors of the matrix elements on the second row of the matrix $A = \begin{pmatrix} -3 & 2 & 3 \\ 1 & 3 & 6 \\ 0 & 7 & 1 \end{pmatrix}$.

Problem 2. Find the adjoint matrix of $A = \begin{pmatrix} -3 & 2 & 3 \\ 1 & 3 & 6 \\ 0 & 7 & 1 \end{pmatrix}$. Then find all non-diagonal elements of the matrix product $A \times adjA$. Explain why they are equal to zero.

Problem 3. Find the inverse of $A = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$. Check the result by definition.

Problem 4. Let $A = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -6 \\ 1 & 7 \end{pmatrix}$. Solve for *X* the matrix equation $AX = B$. Verify solution.

Problem 5. Let $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 8 \\ 1 & 7 \end{pmatrix}$. Solve for *X* the matrix equation $XA^2 = B$. Verify solution.

Problem 6. Evaluate the inverse of $A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}$. Find the product $A^{-1}A$ to check up the result.

Problem 7. Evaluate the inverse of $A = \begin{pmatrix} 1 & 5 \\ -2 & -7 \end{pmatrix}$ by means of the elementary transformations of the extended matrix $(A|I)$.

1.4 Systems of Linear Equations

An $m \times n$ matrix *A* is said to be the matrix of **rank** *r*, if there exists at least one regular submatrix of order r ; every submatrix of a higher order is singular.

According to the definition, rank $A \leq \min\{m, n\}$.

The rank of a matrix can be evaluated by applying just those elementaryrow and column operations which are used to simplify determinants, that is,

1) Interchanging two rows or columns.
2) Multiplying a row (column) by a nonzero number.
3) Multiplying a row (column) by a number and adding the result toanother row (column).

If a row or column consists of zeros then it can be omitted.

These operations are said to be **elementary transformations** of a matrix.

Elementary transformations of the linear system is the process of obtaining an equivalent linear system from the given system by the following operations:

1) Interchange of two equations.
2) Multiplication of an equation by a nonzero number.
3) Addition of an equation multiplied by a constant to another equation.

Each of the above operations generates an equivalent linear system.

Two linear systems of equations are equivalent if one of them can be obtained from another by the elementary transformations.

Problem 1. Reduce the matrix *A* to the row echelon form and find the rank of *A*.

$$\mathrm{A} = \begin{pmatrix} 3 & -4 & 1 & 5 & -2 \\ 2 & 1 & -3 & 0 & 4 \\ 3 & 7 & -10 & -5 & 14 \end{pmatrix}.$$

Problem 2. Find the rank of $\mathrm{A} = \begin{pmatrix} 3 & -4 & 1 & 5 \\ 2 & 1 & -3 & 0 \\ 3 & 7 & -10 & -5 \\ 3 & -1 & 8 & -2 \end{pmatrix}$ by elementary transformations.

Hint: You can interchange two rows or columns, multiply a row or column by a nonzero number and multiply a row (column) by a number to add the result obtained to another row (column).

Problem 3. Solve the system of linear equations via Gaussian elimination.

Check whether the solution satisfies all the given equations.

$$\begin{cases}5x_1 + 2x_2 - 4x_3 = 4,\\ x_1 + 3x_2 + 4x_3 = 3,\\ 2x_1 + 4x_2 + x_3 = 1.\end{cases}$$

Problem 4. Use Gaussian elimination to solve the system of equations

$$\begin{cases}7x_1 + x_2 - 5x_3 = 3,\\ 4x_1 + x_2 - 3x_3 = 5,\\ -2x_1 + x_2 + x_3 = 2.\end{cases}$$

Problem 5. Find the general solution and a particular solution of the linear system, which is given by the augmented matrix

$$\left(\begin{array}{cccc|c} -3 & -2 & 2 & 1 & 2\\ 2 & 1 & -1 & 1 & -2\\ -1 & 1 & -3 & 1 & 4\end{array}\right).$$

Check the solution by substituting the values of the unknowns.

Problem 6. Let $A = \begin{pmatrix} -2 & 2 & 5 & 3\\ 2 & -2 & 1 & 1\\ 5 & -5 & -2 & 4\end{pmatrix}$ be the coefficient matrix of the homogeneous system of linear equations $AX = 0$.Find the general solution.

Problem 7. The homogeneous system of linear equations is given the coefficient matrix $A = \begin{pmatrix} 2 & 1 & -5\\ -3 & -1 & 2\\ 4 & 1 & 0\end{pmatrix}$. Determine the number of solutions.

Problem 8. Use Cramer’s Rule to solve the following system of linear equations.

$$\begin{cases}5x_1 + 2x_2 - 4x_3 = 4,\\ x_1 + 3x_2 + 4x_3 = 3,\\ 2x_1 + 4x_2 + x_3 = 1.\end{cases}$$

Problem 9. Given the system of linear equations

$$\begin{cases}x_1 + 2x_2 + 3x_3 = a,\\ 4x_1 + 5x_2 + 6x_3 = b,\\ 7x_1 + 8x_2 + 9x_3 = c.\end{cases}$$

For which values of *a*, *b*, and *c* the system is consistent?

Hint: Transform the augmented matrix to the reduced row echelon form andthen apply Cramer's General Rule.

Problem 10. Find the values of a for which the following system of linear equations

$$\begin{cases}2x_1 + 3x_2 + x_3 = 3,\\ -x_1 + x_2 = 2,\\ x_1 + ax_2 - x_3 = -2.\end{cases}$$

has the unique solution.

Problem 11. Given the reduced row echelon form of the augmented matrix, $\bar{A} =$

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 5 & 2 \\ 0 & 7 & 0 & 2 & 4 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Find the number of solutions of the corresponding linear system. It is not necessary to solve the system.

Independent work

Determinants

1. Calculate the second-order determinant

a) $\begin{vmatrix} 1 & 2 \\ -3 & -4 \end{vmatrix}$; b) $\begin{vmatrix} x & xy \\ 1 & y \end{vmatrix}$; c) $\begin{vmatrix} 4 & -6 \\ 2 & -3 \end{vmatrix}$.

2. Calculate the third-order determinant

a) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 4 & 6 & 7 \end{vmatrix}$; b) $\begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{vmatrix}$.

3. Calculate the third-order determinant in two ways

a) $\begin{vmatrix} 5 & 6 & 3 \\ 0 & 2 & 0 \\ 7 & -4 & 5 \end{vmatrix}$; b) $\begin{vmatrix} 3 & 0 & 2 \\ -5 & 3 & -1 \\ 6 & 0 & 3 \end{vmatrix}$.

4. Calculate the determinant using row or column decomposition

a) $\begin{vmatrix} 2 & 1 & -3 \\ 0 & 1 & -1 \\ 3 & -2 & 1 \end{vmatrix}$; b) $\begin{vmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 0 & 8 \\ 3 & 0 & 0 & 2 \\ 4 & 4 & 7 & 5 \end{vmatrix}$; c) $\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$.

5. Solve the equation

a) $\begin{vmatrix} x+3 & x-1 \\ 7-x & x-1 \end{vmatrix} = 0$; b) $\begin{vmatrix} 2 & 0 & 3 \\ -1 & 7 & x-3 \\ 5 & -3 & 6 \end{vmatrix} = 0$.

Matrices

1. Find the matrices $A+B$, $A–B$

a) $A = \begin{pmatrix} 4 & 58 & 0 \\ 5 & 22 & -3 \\ 10 & 4 & 17 \end{pmatrix}$, $B = \begin{pmatrix} 13 & 13 & 10 \\ -8 & 4 & 6 \\ 7 & -35 & -8 \end{pmatrix}$;

b) $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 4 & 4 \\ 3 & 5 & 6 \\ 2 & -1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 5 & 4 \\ -5 & 22 & 7 \\ 1 & 0 & 4 \\ 0 & -1 & 2 \end{pmatrix}$.

2. Find the product of the matrix by the number

a) $2 \cdot \begin{pmatrix} 3 & 4 \\ -3 & 8 \end{pmatrix}$;   b) $\frac{1}{3} \cdot \begin{pmatrix} 3 & 81 & 45 \\ 27 & 36 & 90 \\ 15 & 12 & 9 \end{pmatrix}$.

3. Find the product of the matrices A and B, if they exist

a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ -2 & 2 \\ -1 & 0 \end{pmatrix}$;

b) $A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$;

c) $A = \begin{pmatrix} 4 & 0 & -2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 5 \\ 2 \end{pmatrix}$;

d) $A = \begin{pmatrix} 2 & 4 & -1 \\ 3 & 0 & -2 \\ 7 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 4 & 3 \\ 1 & -3 & 1 \end{pmatrix}$;

4. Check whether the matrices are permutable

a) $A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$;

b) $A=\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$; $B=\begin{pmatrix} -1 & -3 \\ 1 & -2 \end{pmatrix}$.

5. Transpose the matrices

a) $A=\begin{pmatrix} 3 & 0 \\ 2 & -5 \end{pmatrix}$; b) $A=\begin{pmatrix} 1 & 0 \\ -3 & 2 \\ 5 & -1 \end{pmatrix}$.

6. Reduce the matrix to a stepwise form

a) $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & -4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 6 & -1 \\ 4 & -9 & 5 \\ -2 & 3 & 5 \\ 1 & 3 & 2 \end{pmatrix}$.

7. Find the matrix A^n

a) $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; b) $A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

8. Find the rank of the matrix

a) $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 1 & 3 & -7 & 1 \\ 2 & -1 & 1 & 6 & -4 \\ -1 & 2 & -1 & -10 & 5 \\ 2 & -1 & 2 & 5 & -4 \end{pmatrix}$.

9. Find the inverse matrix

a) $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$.

10. Solve the matrix equation

a) $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} X = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$;

b) $X \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

Systems of Linear Equations

1. Solve a system of linear equations using the Gauss method

a) $\begin{cases} x_1 - x_2 = -4, \\ 2x_1 + x_2 = -5 \end{cases}$

b) $\begin{cases} x_1 + 2x_2 + 3x_3 = 6, \\ 4x_1 + 5x_2 + 6x_3 = 9, \\ 7x_1 + 8x_2 = -6 \end{cases}$

c) $\begin{cases} 2x_1 - x_2 + 4x_3 = 7, \\ 7x_1 + 3x_2 - x_3 = 3, \\ 5x_1 - 2x_2 - 3x_3 = 4. \end{cases}$

d) $\begin{cases} 2x_1 - 3x_2 + x_3 = -7, \\ x_1 + 2x_2 - 3x_3 = 14, \\ -x_1 - x_2 + 5x_3 = -18 \end{cases}$

e) $\begin{cases} 2x_1 + 3x_2 - x_3 = 4, \\ x_1 + 2x_2 + 2x_3 = 5, \\ 3x_1 + 4x_2 - 5x_3 = 2. \end{cases}$

f) $\begin{cases} 4x_1 + 2x_2 - x_3 = 0, \\ x_1 + 2x_2 + x_3 = 1, \\ x_2 - x_3 = -3. \end{cases}$

2. Solve a system of linear equations using Kramer's formulas

a) $\begin{cases} 2x_1 + 3x_2 + 8x_4 = 0, \\ x_2 - x_3 + 3x_4 = 0, \\ x_3 + 2x_4 = 1, \\ x_1 + x_4 = -24 \end{cases}$

b) $\begin{cases} 2x_1 + x_3 + 3x_4 = 0, \\ -x_1 + 2x_2 - x_3 - 2x_4 = 2, \\ x_1 - x_2 + 4x_4 = 2, \\ x_1 + 2x_2 + x_3 + 3x_4 = 3. \end{cases}$

c) $\begin{cases} 2x_1 - 3x_2 - x_3 + 6 = 0, \\ 3x_1 + 4x_2 + 3x_3 + 5 = 0, \\ x_1 + x_2 + x_3 + 2 = 0 \end{cases}$

d) $\begin{cases} x_2 + 3x_3 + 6 = 0, \\ x_1 - 2x_2 - x_3 = 5, \\ 3x_1 + 4x_2 - 2x_3 = 13 \end{cases}$

3. Find a particular solution to a system of linear equations

a) $\begin{cases} 4x_1 - 3x_2 + 2x_3 = 9, \\ 2x_1 + 5x_2 - 3x_3 = 4, \\ 5x_1 + 6x_2 - 2x_3 = 18 \end{cases}$

b) $\begin{cases} x_1 + x_2 - x_3 = -4, \\ x_1 + 2x_2 - 3x_3 = 0, \\ -2x_1 - 2x_3 = 16 \end{cases}$

4. Solve homogeneous system of linear equations

a) $\begin{cases} x_1+x_2+x_3=0, \\ 3x_1-x_2-x_3=0, \\ 2x_1+3x_2+x_3=0. \end{cases}$ b) $\begin{cases} 3x_1-4x_2+x_3-x_4=0, \\ 6x_1-8x_2+2x_3+3x_4=0. \end{cases}$

2 Vector algebra

2.1 Linear Vector Operations

$$\mu\vec{a} = \{\mu\overrightarrow{a_1}, \mu\overrightarrow{a_2}, \mu\overrightarrow{a_3}\}$$
$$\vec{a} + \vec{b} = \{\overrightarrow{a_1} + \overrightarrow{b_1}, \overrightarrow{a_2} + \overrightarrow{b_2}, \overrightarrow{a_3} + \overrightarrow{b_3}\}$$

Figure 1 ─ Basic formulas

Problem 1. Given two vectors $\vec{a} = \{4,1,-2\}$ and $\vec{b} = \{1,5,-2\}$ in Cartesian coordinate system.

Find the lengths of the vectors $\vec{p} = \vec{a} + \vec{b}$, $\vec{q} = \vec{a} - \vec{b}$.

Problem 2. Let $\vec{a} = \{4,1,-2\}$ and $\vec{b} = \{1,5,-2\}$. Solve for $\vec{p}$ the following vector equation: $3\vec{a} - 2\vec{p} = 4\vec{b}$.

Problem 3. Let $A(1,2,3)$ be the common origin of two vectors $\vec{a} = \overrightarrow{AB} = \{-2,4,7\}$ and $\vec{b} = \overrightarrow{AC} = \{3,-1,5\}$. Find the coordinates for the points B, C.

Problem 4. Find the unit vector $\vec{u}$ in the direction of $\overrightarrow{AB}$, if $A(5,3,-1)$ and $B(2,7,-1)$.

Problem 5. Let $\vec{a} = \{-3,4,0\}$. Find the unit vector $\vec{u}$ in the direction of $\vec{a}$.

In Problems 6 through 8 determine whether the given vectors $\vec{a}$ and $\vec{b}$ are parallel.

Problem 6. $\vec{a} = \{-2,4,5\}$ and $\vec{b} = \{8,16,-20\}$.

Problem 7. $\vec{a} = \{3,4,5\}$ and $\vec{b} = \{6,7,8\}$.

Problem 8. $\vec{a} = \{3,-5,1\}$ and $\vec{b} = \{6,-10,2\}$.

Problem 9. Let A, B, C, D be the vertices of a parallelogram. Express vectors $\overrightarrow{d_1} = \overrightarrow{AC}$ and $\overrightarrow{d_2} = \overrightarrow{BD}$ as linear combinations of two adjacent vectors $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AD}$.

Problem 10. Assume that the vectors $\overrightarrow{d_1} = \overrightarrow{AC}$ and $\overrightarrow{d_2} = \overrightarrow{BD}$ join the opposite vertices of a parallelogram. Find the adjacent vectors $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AD}$.

Problem 11. Given a triangle with the vertices at the points $A(2,1,-1)$, B(3,4,4), $C(5,2,3)$, find: 1) the medians AD, BE and CF; 2) the coordinates of the medians interception point.

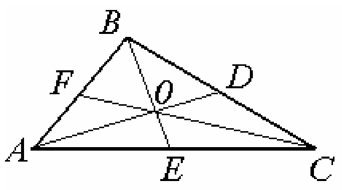

Figure 2 — Drawing of problem 11

Problem 12. Let $A(2,1,-1)$ and $B(8,-2,11)$ be the endpoints of a linear segment AB. Find the coordinates of the point C dividing AB in the ratio 2:1.

Figure 3 — Drawing of problem 12

Problem 13. Consider two vectors $\vec{a} = \{-2,4.5\}$ and $\vec{b} = \{x,-3.z\}$. For which values of z the equation $|\vec{a}| = |\vec{b}|$ has exactly one solution with respect to x? For which values of z the equation has exactly two solutions with respect to x?For which values of z the equation for x has no solutions?

2.2 Scalar Product

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos\varphi$$
$$\text{Proj}_{\vec{b}}\vec{a} = \vec{a}\cos\varphi$$

Figure 4 — Basic formulas

Problem 1. Find the scalar product of two vectors $\vec{a} = \{2,-3,4\}$ and $\vec{b} = \{7,5,1\}$.

Problem 2. Let $|\vec{a}| = 3$, $|\vec{b}| = 4$, and the angle between vectors $\vec{a}$ and $\vec{b}$ equal to $60°$. Find the scalar product $\vec{a} \cdot \vec{b}$.

Problem 3. Let $\vec{a} = 4\vec{p} - 3\vec{q}$ and $\vec{b} = \vec{p} + 2\vec{q}$. Find the scalar product $\vec{a} \cdot \vec{b}$, if $|\vec{p}| = 5$, $|\vec{q}| = 2$, and vectors $\vec{p}$ and $\vec{q}$ from the angle of $30°$.

In Problems 4 through 6 find the angle between two vectors $\vec{a}$ and $\vec{b}$.

Problem 4. $\vec{a} = \{2,-3,4\}$ and $\vec{b} = \{-1,5,-4\}$.

Problem 5. $\vec{a} = \{0,3,4\}$ and $\vec{b} = \{2,4,-4\}$.

Problem 6. $\vec{a} = \{4,1,-1\}$ and $\vec{b} = \{0,3,7\}$.

In Problems 7 through 9 find the projection of $\vec{a}$ and $\vec{b}$.

Problem 7. $\vec{a} = \{6,5,-1\}$ and $\vec{b} = \{2,-3,4\}$.

Problem 8. $\vec{a} = \{1,-1,5\}$ and $\vec{b} = \{4,3,0\}$.

Problem 9. $\vec{a} = \{-2,-1,6\}$ and $\vec{b} = \{2,5,1\}$.

In Problems 10 through 14 find the find the value of a parameter p to satisfy the required condition.

Problem 10. $\vec{a}=\{6,5,-1\}$ and $\vec{b}=\{p,-3,4\}$, $\vec{a}\perp\vec{b}$.

Problem 11. $\vec{a}=\{2,p,-3\}$ and $\vec{b}=\{3,2,-5\}$, $\vec{a}\perp\vec{b}$.

Problem 12. $\vec{a}=\{2,p,-3\}$ and $\vec{b}=\{10,-5,-15\}$, $\vec{a}||\vec{b}$.

Problem 13. $\vec{a}=\{-6,2,5\}$ and $\vec{b}=\{3,-1,p\}$, $\vec{a}||\vec{b}$.

Problem 14. $\vec{a}=\{1,2,-3\}$ and $\vec{b}=\{4,5,p\}$, $\vec{a}||\vec{b}$.

Problem 15. Let $\vec{a}=\{4,-2,1\}$. Fund the direction cosines.

2.3 Vector Product

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|sin\theta$$

Figure 5 — Basic formulas

Problem 1. Find the vector product of two vectors $\vec{a}=\{2,-3,4\}$ and $\vec{b}=\{7,5,1\}$.

Problem 2. Let $|\vec{a}|23$, $|\vec{b}|=7$, and the angle between vectors $\vec{a}$ and $\vec{b}$ equal to $30°$. Find the absolute value of the vector $\vec{a}\times\vec{b}$.

Problem 3. Simplify the vector product $(2\vec{a}-3\vec{b})\times(\vec{a}+4\vec{b})$.

Problem 4. Let $A(2,3,-2)$, B$(1,7,-1)$, $C(3,4,5)$ be three adjacentvertices of a parallelogram. Find the area S of the parallelogram.

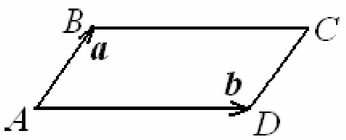

Figure 6 — Drawing of problem 4

Problem 5. Find the area of the triangle with the vertices at the points $A(5,6,2)$, B$(4,8,7)$, $C(5,3,4)$.

Problem 6. Given two vectors $\vec{a}=\{1,-5,4\}$ and $\vec{b}=\{2,-2,1\}$, find a vector $\vec{c}$ such that $\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$.

Problem 7. Given the parallelogram with the adjacent vertices at the points $A(-1,3,2)$, B$(4,1,2)$, $C(3,4,4)$ find the length of the height from the vertex B to the base AC.

Problem 8. Given the triangle with the vertices at the points $A(-1,3,2)$, B$(4,1,2)$, $C(3,4,4)$, find the length of the height from the vertex B to the base AC.

2.4 Scalar Triple Product

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Figure 7 ─ Basic formula

Problem 1. Determine whether the vectors $\vec{a} = \{-3,4,-2\}$, $\vec{b} = \{0,-3,2\}$ and $\vec{c} = \{2,1,-1\}$ are linear independent.

Problem 2. Determine whether the vectors $\vec{a} = \{1,5,-2\}$, $\vec{b} = \{4,-1,3\}$ and $\vec{c} = \{2,-2,1\}$ form a basis.

Problem 3. Determine whether four points $A(2,-3,1)$, $\mathrm{B}(3,3,0)$, $C(3,1,4)$ and $\mathrm{D}(5,-4,1)$ lie in the same plane.

Problem 4. Find the volume *V* of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as it is shown in the figure.

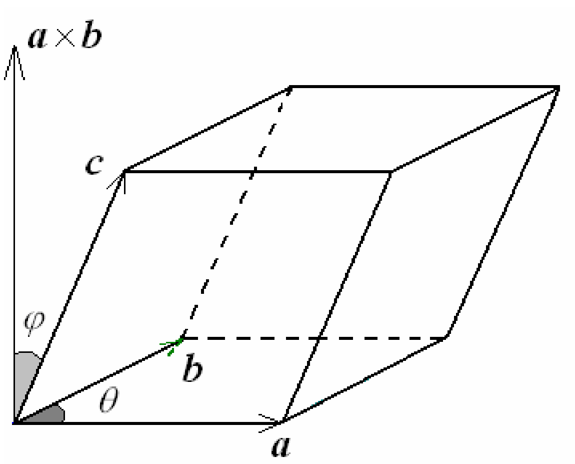

Figure 8 ─ Drawing of problem 4

Problem 5: The tetrahedron is given by the vertices $A(0,-1,1)$, $\mathrm{B}(2,0,3)$, $C(0,4,1)$ and $\mathrm{D}(3,3,3)$. Find the height from the point *D* to the base *ABC*.

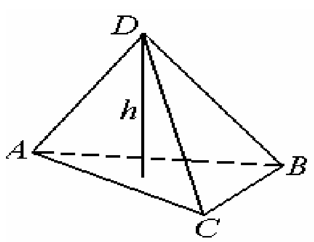

Figure 9 ─ Drawing of problem 5

Problem 6: Find the volume *V* of the tetrahedron with the vertices at the points $A(3,-1,4)$, $\mathrm{B}(4,4,4)$, $C(1,0,2)$ and $\mathrm{D}(1,5,2)$.

Independent work

1. Find the length of the vector $\vec{a}=(1,2,2)$.
2. Find the coordinates of the vector $\vec{AB}$, if $A(1,3,2)$, $B(7,4,2)$.
3. Find the scalar product of the vectors and the cosine of the angle between them

   а) $\vec{a}=(3,4,7)$, $\vec{b}=(1,-7,-2)$, б) $\vec{a}=(2,2,3)$, $\vec{a}=(4,9,-1)$.

4. The vectors $\vec{a}=(m,3,5)$ and $\vec{b}=(2,m,-5)$ are given. At what value is m $\vec{a}\perp\vec{b}$?
5. Find the vector products of the vectors $\vec{a}=(2,3,5)$, $\vec{b}=(1,2,1)$.
6. Calculate the area of the parallelogram built on the vectors $\vec{a}=(6,3,-2)$, $\vec{b}=(3,-2,6)$.
7. Calculate the area of a triangle with vertices $A(1,1,1)$, $B(2,3,4)$, $C(4,3,2)$.
8. Calculate the area of the parallelogram built on the vectors $\vec{a}+3\vec{b}$ и $3\vec{a}+\vec{b}$, if $|\vec{a}|=|\vec{b}|=1$, $(\vec{a}\wedge\vec{b}) = 30°$.
9. Find the product $(5\vec{a}+3\vec{b})(2\vec{a}-\vec{b})$, if $|\vec{a}|=2$, $|\vec{b}|=3$, $\vec{a}\perp\vec{b}$.
10. Find the guiding cosines $\vec{a} = (1,5,-15)$.
11. $|\vec{a}_1|=4$, $|\vec{a}_2|=3$, $\angle(\vec{a}_1,\vec{a}_2)=\frac{\pi}{6}$. Calculate $|\vec{a}_1\times\vec{a}_2|$.
12. Find the vector products of the vectors $\vec{a}=(1,1,4)$, $\vec{b}=(3,2,1)$.
13. Calculate the area of the parallelogram built on the vectors $3\vec{a}+2\vec{b}$ и $2\vec{a}-\vec{b}$, если $|\vec{a}|=4$, $|\vec{b}|=2$, $(\vec{a}\wedge\vec{b}) = 270°$.
14. $|\vec{a}|=5$, $|\vec{b}|=1$, $(\vec{a},\vec{b})=3$. Calculate $|\vec{a}\times\vec{b}|$.

3 Analytical geometry

3.1 Straight Lines

$$\frac{x-x_0}{q_x}=\frac{y-y_0}{q_y}=\frac{z-z_0}{q_z}$$
$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}$$
$$cos\theta=\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}=\frac{p_xq_x+p_yq_{y+}p_zq_z}{\sqrt{p_x^2+p_y^2+p_z^2}\cdot\sqrt{q_x^2+q_y^2+q_z^2}}$$

Figure 10 ─ Basic formulas

Problem 1. Find the canonical equations of the line passing through the point $M(2,-3,4)$ and being parallel to the vector $\vec{a}=\{5,-1,7\}$.

Problem 2. Find equations of the line passing through the point $M(4,0,8)$ and being parallel to the vector $\overrightarrow{AB}$, if $A(-3,2,6)$ and B$(1,4,-4)$.

Problem 3. Find equations of the line passing through the point $M(0,-1,3)$ and being parallel to the line $\frac{x-3}{2}=\frac{y+5}{1}=\frac{z-7}{-4}$.

Problem 4. Let L be a line passing through the points $M_1(4,-1,3)$ and $M_2(3,5,-2)$. Determine whether the point $A(1,3,6)$ lies on the line L.

Problem 5. Let L be a line passing through the points $M_1(4,-1,3)$ and $M_2(3,5,-2)$. Find a few other points on the line L.

Problem 6. In the x,y −plane a line is given by the equation $2x-3y+24=0$. Find 1) any two points on the line; 2) the slope of the line; 3) the *x*-intercept and *y*-intercept.

Problem 7. In the x,y −plane, find the equation of the line passing through the point $M_1(2,-4)$ and being perpendicular to the vector $\vec{n}=\{3,1\}$.

Problem 8. Let $M_1(1,5)$ and $M_2(-2,3)$ be the points on a line. Which of the following points $A(6,4)$, B$(2,7)$ and$C(2,10)$ are the points on the line?

Problem 9. Let $3x-2y+8=0$ and $x+4y-5=0$ be two lines in the x,y −plane. Find the cosine of the angle between the lines.

Problem 10. Transform the equation of the line $x+4y-5=0$ in x,y −plane to the intercept form.

Problem 11. Let $A(2,-1)$, B$(4,4)$, $C(9,7)$ be the vertices of a triangle. Find the equation of the altitude from the vertex *A*. Write down the equation in the intercept form.

Problem 12. Find the distance from the point $M(-2,5)$ to the line $4x-3y+1=0$.

Problem 13. Let ABC be a triangle with the vertices at the points $A(3,5)$, B$(2,-1)$, $C(6,7)$ in the x,y −plane.

Find the length of the altitude from the vertex A.

Problem 14. Find the point of intersection of the lines $3x-2y+8=0$ and $x+4y-5=0$.

Problem 15. Find the point of intersection of the lines $x-2y-3=0$ and $-3x+6y+5=0$.

3.2 Planes

$$\mathrm{Ax+By+Cz+D=0}$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & y_3-y_1 \end{vmatrix} = 0$$

$$cos\theta = \frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}$$

Figure 11 — Basic formulas

Problem 1. Find the general equation of the plane passing through the point $M(5,-2,1)$ and being parallel to the vectors $\vec{a}=\{-4,3,0\}$ and $\vec{b}=\{1,-6,2\}$.

Problem 2. Find the general equation of the plane passing through the points $M_1(-1,4,2)$, $M_2(3,4,-1)$ and $M_3(0,5,6)$.

Problem 3. Find the general equation of the plane passing through the point $M(1,-2,-3)$ and being perpendicular to the $\vec{n}=\{7,-4,-1\}$.

Problem 4. A plane is given by the equation $x-2y+3z-6=0$. Find a unit normal vector $\vec{u}$ to the plane and any two points in the plane.

Problem 5. Transform the equation of the plane $3x+2y-4z-24=0$ to the intercept form.

Problem 6. Find the point of intersection of the line $\frac{x-1}{2}=\frac{y+3}{-2}=\frac{z}{5}$ and the plane $3x-y+2z=4$.

Problem 7. Let L be the line of interception of two planes $2x-y+4z=5$ and $x+y-2z=6$. Find the canonical equations of the line L.

Problem 8. Find the angle between two planes $-3x+4y-z=5$ and $2x+3y-1=0$.

Problem 9. Find the angle between the plane $4x+3y-5z+2=0$ and the line $\frac{x+3}{7}=\frac{y-2}{4}=\frac{z+5}{-1}$.

Problem 10. Find the distance from the point $M(-2,7,-1)$ to the plane $4x-3y+5=0$.

Problem 11. Find the point of intersection of the planes $x-2y+z+5=0$, $3x+y-z+3=0$ and $x-2z+1=0$.

3.3 Quadratic Curves

$$(x-x_0)^2+(y-y_0)^2=R^2$$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=\pm1$$

$$(y-y_0)^2=\pm2p(x-x_0)$$

$$(x-x_0)^2=\pm2p(y-y_0)$$

Figure 12 – Basic formulas

Problem 1. Sketch the graph of $x^2-4x+y^2+6y-3=0$.

Problem 2. Given the circle $x^2-4x+y^2+6y-3=0$, find the radius and coordinates of the center.

Problem 3. Write down the equations of the circle with center at the point $M(-3,5)$ and radius 2.

Problem 4. Reduce each of the following equations to the canonical form:

1) $3x^2-12x+2y^2+4y=11$;
2) $3x^2-12x+2y^2+4y=-14$;
3) $2x^2-4x+3y^2+12y=-15$;
4) $x^2-6x-2y^2+8y=3$;
5) $2x^2+4x-3y^2-12y=0$;
6) $x^2+2x-4y^2=-1$.

Give the detailed description of the curves.

Problem 5. $\frac{(x-5)^2}{25}+\frac{(y+3)^2}{9}=1$. Using the equation recognize the curve and find the location of the center, the major axis and minor axis, the coordinates of the focuses, the eccentricity.

Problem 6. $\frac{(x+2)^2}{9}-\frac{(y-1)^2}{16}=1$. Using the equation recognize the curve and find the location of the center, the coordinates of the focuses, the eccentricity of the hyperbola, the equations of the asymptotes.

4 Control works

Control work № 1

Option № 1

Task 1. Calculate the determinant

$$a)\quad \Delta=\begin{vmatrix}8 & 6\\ 5 & -1\end{vmatrix},\quad b)\quad \Delta=\begin{vmatrix}1 & 2 & 4\\ -2 & 2 & -3\\ 2 & 9 & 0\end{vmatrix},\quad c)\quad \Delta=\begin{vmatrix}1 & 0 & 1 & 1\\ -1 & 2 & 3 & 2\\ 1 & 1 & 0 & -3\\ 0 & 1 & 3 & -1\end{vmatrix}.$$

Task 2. Find the inverse matrix

$$A=\begin{pmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 2 & 4 & 1\end{pmatrix}.$$

Task 3. Solve the matrix equation

$$\begin{pmatrix}4 & -6\\ 2 & 1\end{pmatrix}\cdot X=\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}.$$

Option № 2

Task 1. Calculate the determinant

$$a)\quad \Delta=\begin{vmatrix}5 & 3\\ -2 & 3\end{vmatrix},\quad b)\quad \Delta=\begin{vmatrix}1 & 2 & -1\\ 3 & 7 & 4\\ 0 & 3 & 4\end{vmatrix},\quad c)\quad \Delta=\begin{vmatrix}1 & 1 & 1 & 2\\ 2 & 2 & 3 & 4\\ 1 & 1 & 6 & 10\\ 1 & 4 & 10 & 7\end{vmatrix}.$$

Task 2. Find the inverse matrix

$$A=\begin{pmatrix}3 & 2 & 1\\ 1 & 1 & 2\\ 3 & 4 & 3\end{pmatrix}.$$

Task 3. Solve the matrix equation

$$X\cdot\begin{pmatrix}-4 & 2\\ 5 & 1\end{pmatrix}=\begin{pmatrix}-4 & 2\\ -2 & 3\end{pmatrix}.$$

Control work № 2

Option № 1

Task 1. Solve a system of linear equations using Kramer's formulas

$$\begin{cases} x+y+z=0, \\ 2x+y=4, \\ x-y-2z=5. \end{cases}$$

Task 2. Solve a system of linear equations using the Gauss method

$$\begin{cases} 3x_1+3x_2+2x_3+5x_4+8x_5=1\ldots \\ x_1+3x_2+x_3+4x_4+5x_5=1\ldots \\ 3x_1+4x_2+2x_3+6x_4+9x_5=1\ldots \\ 3x_1+2x_2+3x_3+5x_4+8x_5=1\ldots \end{cases}$$

Task 3. Find the rank of the matrix

$$\begin{pmatrix} 5 & -3 & -3 & 2 & 5 & 4 & 7 \\ 0 & 1 & 5 & -2 & 5 & 3 & -4 \\ 5 & -2 & 2 & 0 & 10 & 7 & -3\ldots \\ 5 & -1 & 7 & -2 & 15 & 10 & -7 \end{pmatrix}$$

Option № 2

Task 1. Solve a system of linear equations using Kramer's formulas

$$\begin{cases} 2x+y+z=3, \\ 5x-2y+z=0, \\ x+2z=5. \end{cases}$$

Task 2. Solve a system of linear equations using the Gauss method

$$\begin{cases} 3x_1+3x_2+x_3+4x_4+7x_5=1\ldots \\ x_1+3x_2+3x_4+4x_5=7, \\ 3x_1+4x_2+x_3+5x_4+8x_5=1\ldots \\ 3x_1+2x_2+2x_3+4x_4+7x_5=1\ldots \end{cases}$$

Task 3. Find the rank of the matrix

$$\begin{pmatrix} 4 & -3 & -1 & 0 & 2 & 8 & 7 \\ 2 & -1 & 6 & 4 & 5 & 3 & -4 \\ -6 & -4 & -5 & 4 & -3 & 11 & -3 \\ 8 & -5 & 1 & 18 & -8 & 14 & -7 \end{pmatrix}.$$

Control work № 3

Option № 1

The coordinates of the verticesof the pyramid are given [illegible], $A_4(-6;-3;6)$:

1) find the length of the edge A_1A_2;
2) the cosine of the angle between the edges A_1A_2 and A_1A_4;
3) face area $A_1A_2A_3$;
4) face equation $A_1A_2A_3$;
5) the equation of the height omitted from the vertex A_4 to the face $A_1A_2A_3$;
6) pyramid volume $A_1A_2A_3A_4$;
7) complete the drawing.

Option № 2

The coordinates of the verticesof the pyramid are given [illegible]:

1) find the length of the edge A_1A_4;
2) the cosine of the angle between the edges A_1A_2 and A_1A_4;
3) face area $A_1A_2A_4$;
4) face equation $A_1A_2A_4$;
5) the equation of the height omitted from the vertex A_3 to the face $A_1A_2A_4$;
6) pyramid volume $A_1A_2A_3A_4$;
7) complete the drawing.

5 Test tasks

Table 1– Option № 1

№	Questions	answers		
		1	2	3
1	At what value of *m* $\vec{a}(1,m,3)\parallel\vec{b}(m,3,1)$?	3	–3	0
2	If $A(2,4,5)$, $B(6,1,2)$, then $\lvert\vec{AB}\rvert$ is equal to	$\sqrt{41}$	$\sqrt{40}$	$\sqrt{39}$
3	$\vec{a}(1,2,3)$, $\vec{b}(2,-1,1)$ тогда $\vec{a}\times\vec{b}$ равно	$\sqrt{66}$	$5\vec{i}+4\vec{j}-5\bar{k}$	$5\vec{i}-4\vec{j}-5\bar{k}$
4	$\lvert\vec{a}_1\rvert=4$, $\lvert\vec{a}_2\rvert=3$, $\left(\widehat{\vec{a}_1,\vec{a}_2}\right)=\frac{\pi}{6}$	$\lvert\vec{a}_1\times\vec{a}_2\rvert=3$	$\lvert\vec{a}_1\times\vec{a}_2\rvert=6$	$\lvert\vec{a}_1\times\vec{a}_2\rvert=3\sqrt{2}$

Table 2– Option № 2

№	Questions	answers		
		1	2	3
1	At what value of *m* $\vec{a}(1,m,2)\parallel\vec{b}(m,9,6)$?	3	–3	1
2	If $A(2,4,5)$, $B(6,1,2)$, then $\vec{BA}(x,y,z)$, where	$x=-1$, $y=-2$, $z=-6$.	$x=1$, $y=2$, $z=6$	$x=3$, $y=6$, $z=6$
3	$\vec{a}(1,2,3)$, $\vec{b}(2,-1,1)$ then $\vec{a}\cdot\vec{b}$ is equal to	$5\vec{i}-4\vec{j}-5\bar{k}$	5	3
4	The area of a parallelogram constructed on vectors $\vec{a}=(1,2,-2)$, $\vec{b}=(3,1,-2)$ is equal to	$\frac{\sqrt{45}}{2}$	$\sqrt{45}$	$2\sqrt{45}$

6 Questions for preparing for the exam in the discipline «Mathematics»

1. Matrices. Actions on matrices.
2. Determinants and methods of their calculation.
3. Properties of determinants.
4. Minors and algebraic complements.
5. Inverse matrix.
6. The rank of the matrix.
7. Systems of linear algebraic equations (joint, incompatible, definite, indefinite.)
8. Solution of non-degenerate systems of linear equations by matrix method.
9. Kramer's formulas.
10. The Gauss method.
12. Homogeneous systems of linear equations.
13. Vectors. Collinear and coplanar vectors.
14. The operations on vectors.
15. Projection of the vector on the axis.
16. The decomposition of a vector the unit vectors of the coordinate axes. The module of the vector. The guides of the cosines.
17. Actions on vectors defined by their coordinates.
18. Linear dependence of the vector system.
19. Basis.
20. Scalar product of vectors.
21. Vector product of vectors.
22. Mixed product of vectors.
23. Forms of writing complex numbers.
24. Actions on complex numbers.
25. Various equations of a straight line on a plane.
26. The angle between two straight lines. Conditions for parallelism and perpendicularity of two straight lines.
27. Second-order lines on the plane: circle, ellipse.
28. Second-order lines on the plane: hyperbola, parabola.
29. Various equations of the plane in space. Conditions for parallelism and perpendicularity of two planes.
30. Various equations of a straight line in space.

7 Exam tickets

Table 3 – Exam ticket №1

Ryazan Institute (branch) Moscow Polytechnic University	**Exam ticket №1 For the discipline "Mathematics" direction of training __________ semester ______**	"APPROVED" Head of the Department __________________ «__» _________
1. Inverse matrix. The rank of a matrix 2. Second-order lines on the plane: hyperbola, parabola. 3. At what value of m are the vectors and mutually perpendicular? $\vec{a} = m\vec{i} + \vec{j}$ и $\vec{b} = 3\vec{i} - 3\vec{j} + 4\vec{k}$ 4. Solve a system of linear equations using Kramer's formulas $\begin{cases} x + y + z = 0, \\ 2x + y = 4, \\ x - y - 2z = 5. \end{cases}$		

Table 4 – Exam ticket №2

Ryazan Institute (branch) Moscow Polytechnic University	**Exam ticket №2 For the discipline "Mathematics" direction of training _________ semester ____**	"APPROVED" Head of the Department __________________ «__» __________
1. Scalar product of vectors. 2. Homogeneous system of linear equations. 3. At what value *m* $\vec{a}(1, m, 2) \parallel \vec{b}(m, 9, 6)$? 4. Find the rank of the matrix . $\begin{pmatrix} 1 & 3 & 0 & -5 & 7 \\ 2 & -1 & 2 & 1 & -3 \\ 4 & -4 & 0 & 0 & 0 \end{pmatrix}$.		

8 Content of lectures and practical classes

Table 5 – Content of lectures and practical classes

Elements of linear and vector algebra	
Matrices and determinants	Matrices, types of matrices. Actions on matrices. The elementary transformations. Second-and third-order determinants, methods for calculating determinants. Properties of determinants. Minors and algebraic complements. A non-degenerate matrix. The inverse matrix. Rank of the matrix.
Systems of linear algebraic equations	Systems of linear algebraic equations. Homogeneous and inhomogeneous systems of linear equations. The Kronecker-Capelli theorem. Solving systems of linear equations using Cramer's formulas, the Gauss method, and the matrix method.
Vectors and operations on them	Vectors. The length of the vector. Ort of the vector. Collinear and coplanar vectors. Linear operations on vectors. Linear dependence and independence of vectors. basis. The decomposition of vectors in the basis vectors. The guides of the cosines. Scalar and vector products of vectors. Mixed product of vectors.
Elements of analytical geometry	
First and second order lines on the plane	A first-order line on a plane (various equations of a straight line). The angle between two straight lines. Conditions for parallelism and perpendicularity of two straight lines. Distance from a point to a straight line. Second-order lines on the plane: circle, ellipse, hyperbola, parabola.
Plane and straight line in space	A plane in space. General equation of the plane. Normal vector. Equation of the plane in segments on axes. The normal equation of the plane. Distance from the point to the plane. The angle between the planes. Conditions for parallelism and perpendicularity of two planes. Various types of equations of a straight line in space. Mutual arrangement of straight lines in space. The relative position of a straight line and a plane.

Bibliographic list

1 Konev, V.V. Linear Algebra, Vector Algebra and Analytical Geometry. Textbook/V.V. Konev. – Tomsk. TPU Press, 2009. – 112 p.

2 Konev, V.V. Linear Algebra, Vector Algebra and Analytical Geometry. Workbook/V.V. Konev. – Tomsk: TPU Press, 2009. – 38 p.

3 Kuznetsov, L.A. Collection of tasks in higher mathematics. typical calculations. Textbook/ L.A. Kuznetsov. – St. Petersburg, 2005. – 240 p.

4 Postnikov, M.M. Linear algebra. Textbook/M.M. Postnikov. – St. Petersburg, 2009. – 400 p.

Educational publication

Azizyan Inara Artushovna

LINEAR ALGEBRA,
VECTOR ALGEBRA AND ANALYTICAL GEOMETRY

WorkBook

Signed to the press _________. Circulation 5 instances.
Ryazan Institute (branch) of Moscow Polytechnic University
390000, Ryazan, Pravo-Lybedskaya Street, 26/53